Teaching for Improved Procedural Flexibility in Mathematics

Jon R. Star, Harvard Graduate School of Education, jon_star@harvard.edu
Bethany Rittle-Johnson, Kelley Durkin, Mason Shero, Jessica Sommer
bethany.rittle-johnson@vanderbilt.edu, kelley.durkin@vanderbilt.edu, mason.s.shero@vanderbilt.edu, jessica.m.sommer@vanderbilt.edu
Vanderbilt University

Abstract: Flexibility in the use of mathematical procedures – or procedural flexibility – has emerged as an important outcome in educational policy and practice. As interest in procedural flexibility as a learning outcome has increased, researchers have also begun to investigate teacher practices and features of learning environments that appear to promote flexibility. Drawing from video records of algebra instruction, we begin to articulate a more holistic theory of teaching for improved flexibility by describing instructional features that appear to be particularly instrumental in teaching for flexibility. We focus on three complementary and interrelated instructional practices - the use of (a) flexibility-eligible tasks, (b) structured instructional routines that leverage comparison of multiple strategies, and (c) discussion prompts that push students to consider affordances and constraints of the strategies that they have used.

Flexibility in the use of mathematical procedures – or procedural flexibility – has emerged as an important outcome in educational policy and practice (National Research Council, 200l; NCTM, 2014a). As noted in a position paper on procedural fluency from the National Council of Teachers of Mathematics (NCTM, 2014b): "All students need to have a deep and flexible knowledge of a variety of procedures, along with an ability to make critical judgments about which procedures or strategies are appropriate for use in particular situations." Researchers have begun to investigate procedural flexibility in mathematical domains including arithmetic (Blöte, Van der Burg, & Klein, 2001), algebra (e.g., Rittle-Johnson & Star, 2007) and calculus (Maciejewski & Star, 2016). Within this literature, flexibility is defined as knowledge of multiple strategies and the ability to select the most appropriate strategy for a given problem and problem-solving circumstances (e.g., Star, 2005).

As interest in procedural flexibility as a learning outcome has increased, researchers have also begun to investigate teacher practices and features of learning environments that appear to promote flexibility. Evidence to date suggests that particular instructional practices such as comparing and contrasting multiple strategies (Star, Newton et al., 2015) and teacher questioning (Star, Pollack et al., 2015) may be linked to improved flexibility among algebra learners. In addition, the quality of mathematical tasks used in classrooms – as well as teachers' implementation of these tasks – is known to influence student learning outcomes (e.g., Henningsen & Stein, 1997). It follows that particular kinds of mathematical tasks may be especially instrumental in promoting procedural flexibility.

In the present paper, our goal is to begin to articulate a more holistic theory of teaching for improved flexibility. We draw upon video records of practice of secondary mathematics instruction, taken from a larger project that investigated the efficacy of a contrasting cases curriculum in algebra classes. We identified exemplar lessons from teachers who successfully promoted growth in students' flexibility. From these exemplars we describe instructional features that appear to be instrumental in teaching for improved procedural flexibility. We ask: (1) What features of mathematical tasks (and the implementation of such tasks) appear to be linked to improvements in students' flexibility? And (2) what teacher practices appear to be linked to improved student flexibility? Building on prior work, we aim to move beyond the preliminary identification of discrete instructional features to an investigation of the ways that these and other related teacher practices and task features come together in a dynamic classroom learning environment in ways that promote flexibility.

Method

Participants in this study included 16 algebra teachers and their 550 students distributed 25 sections of Algebra I in three high schools and one middle school, all in the northeastern US. Teachers volunteered to participate in a research project that investigated the efficacy of a contrasting cases supplemental instructional algebra curriculum. Teachers were introduced to the curriculum and its associated instructional routines during the one-week, 35-hour professional development institute. In addition, throughout the school year, teachers received brief one-on-one support and feedback based on videotapes of their implementation of the curriculum.

Each teacher was videotaped as they taught with the contrasting cases supplemental algebra curriculum by a professional videographer between 4 and 9 times during the Algebra I course. Videotapes were subsequently analyzed by members of the research team using two coding instruments that had been previously used and validated in a prior project (Star, Pollack et al., 2015), one that focused on teachers' use of core instructional practices deemed integral to the use of the supplemental algebra curriculum and the other specifically targeting several additional features that were hypothesized to be instrumental to promoting procedural flexibility. For the present analysis, the highest scoring lesson videos were examined qualitatively and iteratively by the research team, both to identify specific exemplar episodes and also to consider more holistically the ways that teachers were effectively promoting students' flexibility.

Results

Our analysis to date suggests the following instructional features to be especially instrumental for promoting procedural flexibility: The use of (a) flexibility-eligible tasks, (b) a structured instructional routine that leverages comparison of multiple strategies, and (c) discussion prompts that push students to consider affordances and constraints of the strategies that they have used.

Flexibility-eligible tasks

A flexibility-eligible task (Hästö, Palkki, Tuomela & Star, 2019) is one that can be solved in several different ways, including by both non-standard and standard strategies. In other words, a flexibility-eligible task is one where it is possible for a student to demonstrate flexibility. A defining feature of a flexibility-eligible task is that it is possible to consider whether some strategies for completing the task are better or more appropriate than others. All mathematical tasks can be solved in multiple ways, but in many cases, it may be difficult to determine whether some ways are better than others, even with a rather broad and subjective conceptualization of what features of a strategy might make it 'better' than another. In some situations, the better strategy is the more efficient one. But alternatively, sometimes a better strategy is one that is less likely to lead to error, is most easily remembered or implemented, or is the cleverest. Indeed, among mathematicians, the word 'elegant' has been used to identify the best or most valued strategies, even though this widely used term is difficult to precisely define (Hardy, 1940). Flexibility-eligible tasks are those where one can make a reasonable argument that some strategies for completing the task are better than other strategies that also can be successfully used on the task.

One class of flexibility-eligible tasks from the contrasting cases supplemental curriculum used in the present study are problems where one needs to find the solution(s) to linear systems in two variables. As shown in Figure 1, among the strategies that can be used to solve linear systems are elimination and substitution. Depending on the structure and coefficients of the linear system to-be-solved, as well as one's conception of what 'better' means, it is possible to consider whether elimination or substitution is better for a given problem, as shown in Figure 1.

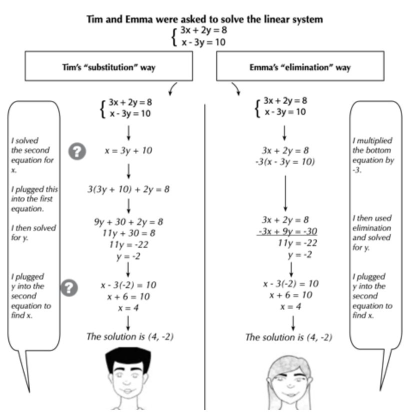

Figure 1. Two strategies for solving linear systems.

Structured instructional routines that leverage comparison

A second instructional feature that we found to be especially instrumental for promoting procedural flexibility is the use of structured instructional routines specifically designed to leverage comparison. Flexibility involves the consideration of multiple strategies; teachers who seek to promote flexibility need instructional routines that allow students to compare and contrast the multiple approaches.

Consider a teacher who wishes to present and discuss multiple strategies for solving a linear systems problem such as the one illustrated in Figure 1. Especially for novice learners, this page – and the consideration of multiple strategies generally – can be a very cognitively demanding task, given the amount of information and text that is present. Teachers who were the most effective in promoting flexibility had developed and frequently used routines that leveraged comparison but that also helped students manage the complexity and density of information that are inherent in comparing and contrasting two complex strategies.

Two examples of these types of effective routines for leveraging comparison were as follows. First, before engaging in a comparison of Tim's and Emma's strategies (see Figure 1), teachers would cover up Emma's strategy and thoroughly discuss Tim's strategy. After discussing Tim's strategy, teachers would then cover up Tim's strategy while discussing Emma's strategy. This cover-up instructional routine allowed students to make sense of each strategy individually before engaging in the more complex and demanding task of comparing the two strategies. And second, teachers would make regular and consistent use of a standard set of prompts for engaging students in comparison. One commonly used prompt that seemed especially effective was, What are some similarities and some differences between Tim's and Emma's ways? This focus on similarities and differences allowed students to ease into the comparison activity by focusing on the similarities and differences in the answers obtained by the two students, their steps, and (in cases where Tim and Emma did not solve the same problem) the problems that were solved.

These two instructional routines – (1) cover up and examine one strategy at a time, and (2) use a standard set of prompts to ease into comparison such as identifying similarities and differences, seemed particularly effective at enabling students to engage in the important but potentially cognitively challenging task of comparing two complex strategies, thus helping to promote flexibility.

Prompts encouraging students to consider affordances and constraints of strategies

A third instructional feature that teachers used to promote procedural flexibility were discussion prompts or questions that focused students' attention to the affordances and constraints of different strategies. A key component of procedural flexibility is the ability to identify the most appropriate strategy for a given problem and some given problem-solving circumstances. Given a flexibility-eligible problem, it was not sufficient to merely compare and contrast multiple strategies – but rather teachers pushed students to think about and discuss which strategies were better and why.

In videos where this type of discussion seemed most productive, note that it was rarely the case that the class was in complete agreement about which strategy was the best one. Rather, effective flexibility-promoting discussions were ones where the students wrestled with the nuance that is inherent in thinking about the affordances and constraints of strategies – much in the same way that mathematicians might engage in debate about the relative elegance of various strategies. And in order to engage with this nuance, teachers needed to pose questions to students that pushed them to consider the pros and cons of each strategy. With respect to the problem illustrated in Figure 1, commonly used prompts included: "For this problem, which method is better?", "What are some advantages of Tim's "substitution" way? Of Emma's "elimination" way?", and "More generally, is there a situation where substitution would be better than elimination, or vice versa?"

Consider the following interactions from one teacher's Algebra I class, as the problem shown in Figure 1 was discussed (all names are pseudonyms). The teacher asks the class, "Which one do you prefer? Which one is better?" – and this prompts a number of students to engage with this question and provide their evaluations of the relative merits of Tim's and Emma's strategies:

Teacher: Which one do you prefer? Which one is better? Jim?
Jim: Uh, probably Emma's way. Less steps.
Teacher: Less steps? I love it. Emma's way, so elimination. Good. Rhonda?
Rhonda: Um, I would do Tim's way just because it makes more sense in my brain.
Teacher: Okay. I love it. Makes more sense in your brain. Nora?
Nora: Same as Rhonda.
Teacher: Same thing, substitution. Okay, so we have two substitution, one elimination. Frank?
Frank: It depends on the situation.
Teacher: Oh, OK. What about this situation? This specific situation. Which one would you do?
Frank: Um. Elimination.
Teacher: Elimination? Wanda?
Wanda: I like substitution. Just because I learned it first.
Teacher: Ah. OK. Yeah, that actually goes a long way. Maybe that has to do with why it makes more sense too, at some point. Sue?
Sue: Probably the substitution.
Teacher: Yeah, nice. José?

José: Um, probably substitution only because I would see that *x* is by itself and then I would just get rid of, like I would just see that and I would know that I don't have to do any extra math, easier.
Teacher: Okay good.
José: Easier for me instead of having to multiply, er, yeah, multiply first, getting rid of one.
Teacher: Yeah, good.

The teacher then moves the discussion forward by posing the question, "So is there a situation where substitution would be better than elimination, or visa-versa?" Students are given a moment to think about this question individually, and then they discuss their ideas with a partner. The class then comes back together and shares their answers. As a culmination of the discussion, the teacher asks students to generate a new system where substitution would be a better strategy, as well as another new linear system where elimination would be a better strategy.

This particular lesson is similar to others where teachers made extensive use of prompts that pushed students to consider the affordances and constraints of multiple strategies. Pushing students to engage in this type of reasoning seemed particularly important for promoting the development of procedural flexibility.

Discussion

In the present analysis, we sought to further our understanding of how teachers can teach for improved procedural flexibility in mathematics. By examining a large corpus of Algebra I video, particularly the teachers whose lessons scored highly on two coding instruments related the use of the a supplemental contrasting cases algebra curriculum, we identified three features of teachers' instructional practice and the tasks used in their lessons that seemed particularly instrumental for the development of flexibility. These features were the reliance on flexibility-eligible tasks, the use of structured instructional routines that effectively leverage comparison, and the use of prompts designed to push students to consider the affordances and constraints of multiple strategies.

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