Comparing and Explaining Examples of Multiple Strategies to Promote Algebra Learning: Instructional Features that Predict Learning

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The purpose of this study was to explore variability in students' algebra learning and instructional features that predict learning when using an example-based instructional intervention. Success in Algebra courses is central for high school graduation, entry into postsecondary education and completion of a STEM degree (see Barnes, Slate, & Rojas-LeBouef, 2010 and Hinojosa et al., 2016 for reviews). The intervention focused on comparing and discussing examples of multiple solution strategies, a recommended instructional method in many countries (Australian Education Ministers, 2006; Brophy, 1999; Kultusministerkonferenz, 2004; NCTM, 2014; Singapore Ministry of Education, 2006; Treffers, 1991). Indeed, expert teachers in the U.S. and Japan have students compare multiple strategies (Ball, 1993; Lampert, 1990; Richland, Zur, & Holyoak, 2007).

Teachers need materials and professional development to effectively use comparison and discussion. U.S. teachers often struggle to effectively support and discuss comparison of multiple strategies in mathematics instruction (Richland, Holyoak & Stigler, 2004; Richland et al., 2007; Stein, Engle, Smith, & Hughes, 2008). For example, high-quality implementation of multiple strategies instruction occurred in only 12% of lesson videos that incorporated this practice (Hill et al., 2014). It is often used merely for encouraging and validating student participation with little mathematical substance (Ball, 2001; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). Further, in a recent textbook analysis, we found that only 3-4% of examples in 2 U.S. Algebra I textbook included multiple strategies for solving *the same problem*, and comparison was not supported.

Our instructional approach, which we now term ***C**omparison and **D**iscussion of **M**ultiple **S**trategies* (*CDMS*), is based on converging evidence from cognitive science and mathematics education research on the importance of (1) knowledge of multiple strategies, (2) comparison, and (3) discussion for mathematics learning**.** Knowing **multiple strategies** allows people to adapt their strategy use to task demands and problem features and is associated with greater understanding (Blöte et al., 2001; Hiebert et al., 1996; Siegler, 1996). Second, **comparison** of multiple strategies is a powerful process that supports learning, including mathematics learning. Comparison improves learning in many domains (see Alfieri, Nokes-Malach, & Schunn, 2013 for a meta-analysis). We have rigorous evidence from our prior work on the benefits of comparison for math learning in the form of short-term classroom studies. Across five studies, having learners compare and discuss multiple strategies led to greater mathematics knowledge than studying and discussing the same strategies sequentially, without comparison (Rittle-Johnson & Star, 2007, 2009; Rittle-Johnson, Star, & Durkin, 2009, 2012; Star & Rittle-Johnson, 2009). Drawing from our research, a Practice Guide from the U.S. Department of Education (Woodward et al., 2012) identified comparing multiple strategies as one of five recommendations for improving mathematical problem solving in the middle grades. Third, the development of mathematics knowledge is enhanced by classroom **discussions** in which students generate explanations and teachers facilitate a discussion of different student responses (Lampert, 1990; Silver et al., 2005; Stein et al., 2008).

Indeed, more frequent engagement by students with other students' strategies and ideas during discussion is related to greater success on a mathematics assessment (Webb et al., 2014).

Teachers using our approach begin by having students compare two strategies presented as worked examples, making sense of each and identifying their similarities and differences. Students then reflect on key points about the comparison, discussing their ideas with a partner and ultimately with the whole class. Materials were developed based on evidence from cognitive psychology and mathematics education (e.g., Begolli & Richland, 2015; Rittle-Johnson & Star, 2007).

Method

Participants. Seven ninth-grade teachers from 3 schools and their 361 students participated. They used our CDMS approach as part of their instruction for 4-5 units during a full-year Algebra I course in the 2017-2018 school year (only one teacher covered the fifth unit). Schools were in suburban Massachusetts and served predominantly white, middle-class students.

Materials and Procedure. All materials are available **my.vanderbilt.edu/cems** under the **Resources tab.** A team of mathematics education experts developed the classroom materials by identifying core concepts, common student difficulties, and key misconceptions, and then creating CDMS materials to attempt to address them. At the core of the supplemental CDMS Algebra I curriculum are worked example pairs (WEPs). Each WEP shows the mathematical work and dialogue of two hypothetical students as they attempt to solve an algebra problem. The curriculum contained four types of WEPs, with the types varying in what is being compared and the instructional goal of the comparison (e.g., Which is correct? Why does it work?). Materials are designed to facilitate comparison, including side-by-side visual presentation of the strategies, spatial cues and common language, and prompts to engage in specific comparisons, which have each been shown to improve learning from comparison (Begolli & Richland, 2015; Catrambone & Holyoak, 1989; Richland et al., 2007). Initial explanation prompts ask students to make sense of steps in individual strategies and identify similarities and differences in the strategies. Subsequent prompts focus on making connections and identifying the big idea of the comparison. A take-away page is included, with the fictitious students identifying the big idea for that WEP. This is intended to help support teachers in summarizing the key points after the class discussion. The current curriculum contains 40 WEPs covering 5 core Algebra I topics, including solving linear equations, functions and graphing linear equations, solving systems of equations and polynomials and factoring

We have further developed and supported a pedagogical routine for supporting small-group work and whole class discussion (Rittle-Johnson, Star, Durkin & Loehr, 2018, 2019). A key discussion question is posed (e.g., "Is there a situation where substitution would be better than elimination, or vice versa?"), and then the class engages in a Think-Pair-Share routine, where students think for a minute on their own, pair with another student to discuss their answers, and then share ideas in a whole-class discussion. At the end of the discussion, the teacher summarizes the main points of the discussion and asks students to write a summary in their own words. This routine is supported by a graphic organizer that prompts students to engage in and take notes on each phase. Teachers are provided with sample questions to help draw out multiple student responses to the same question and to ask students to build on one another's ideas.

Student knowledge was assessed using a researcher-developed algebra assessment (with 30 mostly closed-choice questions tapping conceptual knowledge, procedural knowledge and procedural flexibility of the target 5 units). Teachers administered the test at the beginning and end of the school year. Alpha reliability at posttest was good (alpha = .80). Psychometric analyses (confirmatory factor analysis and latent factor analysis) indicated the measure was best treated as having a single factor.

Teachers reported when they used our materials via a teacher log (paper or online). Classroom lessons were videotaped 2-3 times per target unit by a local videographer, with 6-9 videos coded per teacher. Each 7.5 video segment was coded on a 4 point scale, with 1 indicating low quality and 4 indicating high quality, along 5 dimensions, using a coding scheme adapted from Litke (2019). The current paper focuses on codes for 2 dimensions, using only segments of lessons when our materials were being used (see Table 1). The teacher questioning code identified the highest-level observed, from simple questions (yes/no or calculations) to "why" and open-ended questions (E.g., "What is the answer?" vs. "Can you generate another problem where Riley's method could not be used?"). The student interaction quality code identified the highest level of interaction either between the teacher and students or amongst students observed. We defined interaction as the opportunity to verbally share ideas regarding mathematical procedures and/or content within each lesson segment. Examples of high quality were: "Share with a partner and see if you agree/ disagree and add something that your partner next to you said," or having multiple students respond to the same why question.

Data analyses. Latent transition analysis (LTA) was used to identify student knowledge profiles on the algebra assessment at the beginning of the school year and change from the beginning to the end of the school year. Then, we explored variability between teachers in their students' initial knowledge profile and profile transitions (i.e., change) and evaluated if 3 instructional features (see Table 1) predicted this variability.

Results

Three student knowledge classes were identified in the LTA: students with a low, medium and high level of knowledge. Estimated proportion correct on each item by class is shown in Figure 1. The high-knowledge class was nearly non-existent at pretest (estimated 6% of students), but grew to 50% of students at posttest. The medium-knowledge class was common at pretest (51%) and was much less common at posttest (15%). The low-knowledge class was also common at pretest (43%) and a bit less common at posttest (35%).

There was large variability among teachers in their students' initial knowledge class at pretest as well as in the probability to transition to a higher knowledge class on the end-of-year posttest. As shown in Figure 2, two teachers had few students transition to higher knowledge levels, while other teachers had many students transition.

We explored instructional features that could explain this variability. The higher teachers' use of our materials and the more teachers facilitated high-quality student interactions, the more likely their students were to have a higher-knowledge class at the beginning of the school year (Likelihood Ratio χ^2 (2) = 17.08, $p < .001$ and χ^2 (2) = 21.93, $p < .001$, respectively) and to transition to a higher-knowledge class at the end of the school year (χ^2 (2) = 6.20, $p = .045$ and χ^2 (2) = 18.77, $p < .001$, respectively).

Significance

Greater use of our *CDMS* approach was related to greater knowledge gains, providing preliminary support for the effectiveness of the approach, albeit with a small number of teachers. Greater support for high-quality student interaction was also associated with greater knowledge gains, highlighting the importance of students explaining ideas with classmates. However, some teachers struggled to implement our approach and some students did not learn much of our target content, especially in classrooms with many students with low initial knowledge, suggesting that our CDMS approach and teacher PD was not sufficiently powerful to aid learning by all students.

Based on these findings, we revised some of materials, replacing a few confusing or less useful examples and better integrating discussion questions and big ideas in most materials. We also increased attention to supporting high-quality student interactions in our professional development. The current findings highlight the potential of evidence-based instructional approaches for improving student learning, as well as persistent gaps in improving teaching quality and student learning broadly.

Reflecting on the symposium themes of how, when and with whom to use worked examples to improve mathematics learning, our research suggests:

(1) HOW: Side-by-side presentation of two worked examples, with reflection questions, to scaffold comparison and discussion of the examples

(2) WHEN: Can be used to introduce, expand or review ideas (e.g., beginning, middle or end of lesson)

(3) WITH WHOM: Students with more prior knowledge more easily and reliably benefit from comparing and discussing multiple strategies (see also Rittle-Johnson, Star & Durkin, 2009). Students with little prior knowledge need extra support (see also Rittle-Johnson, Star & Durkin, 2012).

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References

Alfieri, L., Nokes-Malach, T. J., & Schunn, C. D. (2013). Learning through case comparison: A meta-analytic review. *Educational Psychologist, 48*, 87-113. doi: 10.1080/00461520.2013.775712

Australian Education Ministers. (2006). Statements of Learning for Mathematics.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93*, 373-397. doi: 10.1086/461730

Ball, D. L. (2001). Teaching, With Respect to Mathematics and Students. In T. Wood, B. Scott Nelson & J. Warfield (Eds.), *Beyond Classical Pedagogy: Teaching Elementary School Mathematics* (pp. 11-22). New Jersey: Erlbaum.

Barnes, W., Slate, J. R., & Rojas-LeBouef, A. M. (2010). College-readiness and academic preparedness: The same concepts? *Current Issues in Education, 13*. Retrieved from http:// cie.asu.edu/

Begolli, K. N., & Richland, L. E. (2015). Teaching Mathematics by Comparison: Analog Visibility as a Double-Edged Sword. *Journal of Educational Psychology*. doi: 10.1037/edu0000056

Blöte, A. W., Van der Burg, E., & Klein, A. S. (2001). Students' flexibility in solving two-digit addition and subtraction problems: Instruction effects. *Journal of Educational Psychology, 93*, 627-638. doi: 10.1037//0022-0663.93.3.627

Brophy, J. (1999). Teaching. In *Education Practices Series No. 1, International Bureau of Education*. Geneva.

Catrambone, R., & Holyoak, K. J. (1989). Overcoming contextual limitations on problem-solving transfer. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 15*, 1147-1156. doi: 10.1037//0278-7393.15.6.1147

Durkin, K., Rittle-Johnson, B., Star, J. R., & Loehr, A. (under review). Effects of comparing and discussing multiple strategies on students' algebra learning. Paper submitted to the annual meeting of the Society for Research on Educational Effectiveness (SREE), Arlington, VA.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Human, P., Murray, H., . . . Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*, 12-21. doi: 10.2307/1176776

Hill, H.C., Litke, E., Lynch, K.H., Pollard, C. & Gilbert, B. (2014). *Learning lessons from instruction: Descriptive results from an observational study of urban elementary classrooms.* Paper presented at the Association for Public Policy and Management conference, Segovia, Spain.

Hinojosa, T., Rapaport, A., Jaciw, A., LiCalsi, C., & Zacamy, J. (2016). Exploring the foundations of the future STEM workforce: K–12 indicators of postsecondary STEM success (REL 2016–122). Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southwest. Retrieved from http://ies.ed.gov/ncee/edlabs.

Kultusministerkonferenz. (2004). Bildungsstandards im Fach Mathematik für den Primarbereich [Educational Standards in Mathematics for Primary Schools].

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*, 29-63. doi: 10.2307/1163068

Litke, E. (2019). The nature and quality of algebra instruction: Using a content-focused observation tool as a lens for understanding and improving instructional practice. *Cognition and Instruction*. Advance online publication. doi:10.1080/07370008.2019.1616740

NCTM. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: National Council of Teachers of Mathematics, Inc.

Richland, L. E., Holyoak, K. J., & Stigler, J. W. (2004). Analogy use in eighth-grade mathematics classrooms. *Cognitiion and Instruction, 22*, 37-60. doi: 10.1207/s1532690Xci2201_2

Richland, L. E., Zur, O., & Holyoak, K. J. (2007). Cognitive supports for analogies in the mathematics classroom. *Science, 316*, 1128-1129. doi: 10.1126/science.1142103

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*, 561-574. doi: 10.1037/0022-0663.99.3.561

Rittle-Johnson, B., & Star, J. R. (2009). Compared to what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology, 101*, 529-544. doi: 10.1037/a0014224

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2009). The importance of prior knowledge when comparing examples: Influences on conceptual and procedural knowledge of equation solving. *Journal of Educational Psychology, 101*, 836-852. doi: 10.1037/a0016026

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2012). Developing procedural flexibility: Are novices prepared to learn from comparing procedures? *British Journal of Educational Psychology, 82*, 436-455. doi: 10.1111/j.2044-8279.2011.02037.x

Rittle-Johnson, B. Star, J., Durkin, K. & Loehr, A. (2018, May). Comparing solution strategies to promote algebra learning and flexibility. In Hsieh, F. & Kaur, B. (Eds) Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education, Volume 1. Taipei, Taiwan: National Taiwan Normal University.

Rittle-Johnson, B., Star, J., Durkin, K. & Loehr, A. (2019). Compare and discuss to promote deep learning. Manalo, E. (Ed.). *Deeper Learning, Dialogic Learning, and Critical Thinking: Research-Based Strategies for the Classroom.* New York, NY. Routledge.

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking*. New York: Oxford University Press.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior, 24*, 287-301. doi: 10.1016/j.jmathb.2005.09.009

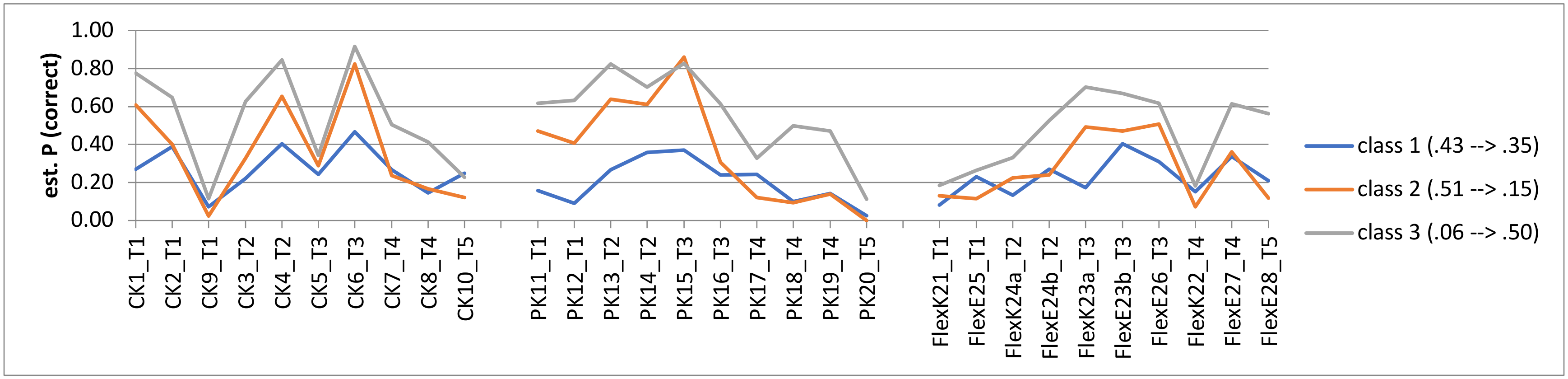
Singapore Ministry of Education. (2006). Secondary Mathematics Syllabuses.
Star, J. R., & Rittle-Johnson, B. (2009). It pays to compare: An experimental study on computational estimation. *Journal of Experimental Child Psychology, 101*, 408 - 426. doi: 10.1016/j.jecp.2008.11.004
Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.
Treffers, A. (1991). Realistic mathematics education in The Netherlands 1980-1990. In L. Streefland (Ed.). *Realistic Mathematics Education in Primary School*. Utrecht: Freudenthal Institute.
Webb, N. M., Franke, M. L., Ing, M., Wong, J., Fernandez, C. H., Shin, N., & Turrou, A. C. (2014). Engaging with others' mathematical ideas: Interrelationships among student participation, teachers' instructional practices, and learning. *International Journal of Educational Research, 63*, 79-93. doi: 10.1016/j.ijer.2013.02.001
Woodward, J., Beckmann, S., Driscoll, M., Franke, M. L., Herzig, P., Jitendra, A. K., & al., e. (2012). *Improving mathematical problem solving in grades 4 to 8: A practice guide*. Washington, D.C.: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.

Table 1
Instructional Features of Algebra Instruction: Proportion of CDMS materials used and ratings of teacher questioning quality and student interaction quality when using our materials.

Teacher ID	Proportion of CDMS Materials Used +	Teacher Questioning Quality Rating*	Student Interaction Quality Rating*
T11	0.41	2.38	3.10
T12	0.63	3.18	2.98
T21	0.78	3.41	3.37
T22	0.67	3.17	2.93
T23	0.54	3.19	2.84
T32	0.65	3.17	2.88
T33	0.54	3.06	3.11
Average	0.61	3.11	3.04

+ Only Teacher 21 covered the 5th targeted unit in her textbook (and thus used our materials for that unit; using the first 2 of 7 CDMS materials for that unit). Other teachers had planned to cover the unit at the end of the school year, but did not have time.
*Each 7.5-minute segment of the lesson using our materials was rated on a 4-point scale, with 1 indicating low quality and 4 indicating high quality. Conceptually-focused and open-ended questions were considered high-quality teacher questioning; meaningful interaction among students in small-group or whole class discussion was considered high-quality student interaction.

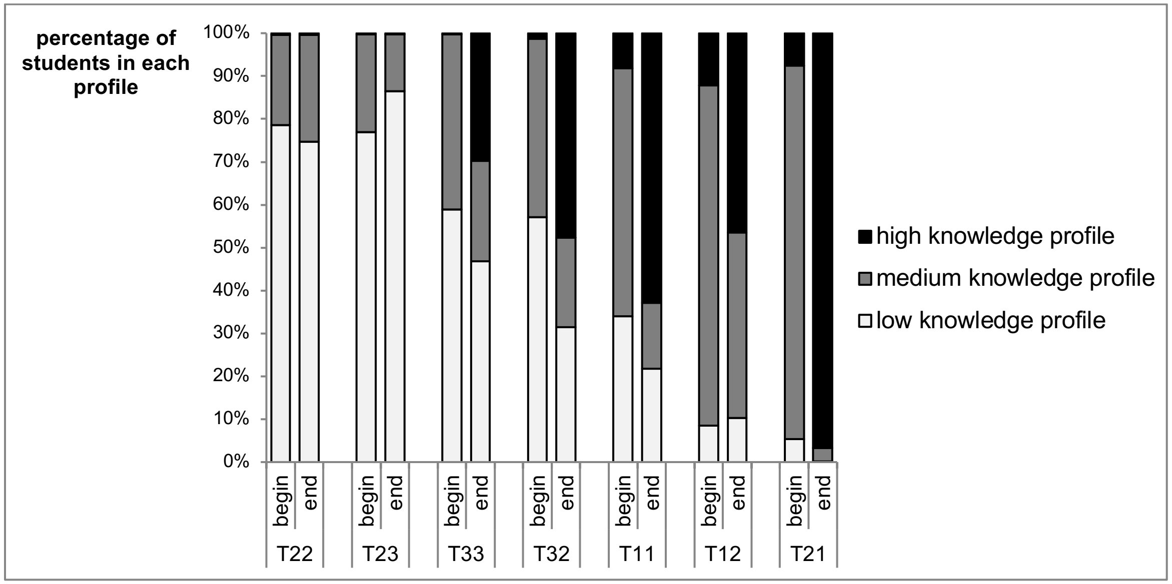
Figure 1
Estimated proportion correct on each item, for the 3 knowledge profiles. Proportion of students estimated to be in each profile at beginning and end of school year are in parentheses.



Notes: Low knowledge (class 1), medium knowledge (class 2) and high knowledge (class 3) profiles. Individual items are along the x-axis, and items are labeled based on targeted knowledge type (CK = Conceptual Knowledge, PK = Procedural Knowledge; Flex = Procedural Flexibility), item number and Topic (e.g., T1 = Topic 1). Performance difference depends on item, e.g. on CK5_T3 difference is rather small, but on PK12_T1 difference is rather large. Moreover, the medium knowledge class performs more like the low knowledge class on some items (e.g. PK18_T4) but more like the high knowledge class on other items (e.g., PK15_T3). The high-knowledge class was nearly non-existent at pretest (6%), but grew to 50% of students at posttest. The medium-knowledge class dropped substantially from pretest (51%) to posttest (15%).

Figure 2

Percentage of students in each knowledge profile at the beginning and at the end of the school year, by teacher



Note: T stands for Teacher, the first digit indicates the school number and the second digit indicates the teacher number at the school. Teachers are ordered by proportion of students in the low knowledge profile at the beginning of the school year.