

Compare and Discuss to Deepen Algebra Learning

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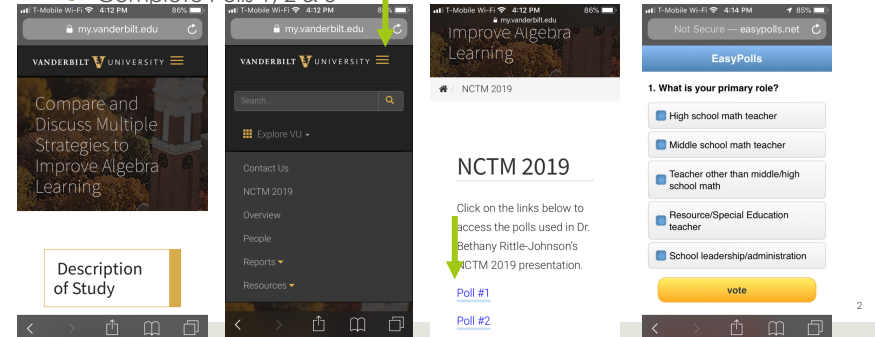


Audience Polls 1 - 3

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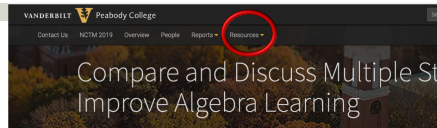
- Select NCTM 2019 from menu

- Complete Polls 1, 2 & 3



From Resources Tab:
Select Materials

All of our materials can be downloaded



Materials

Worked Example Pairs (WEPs)

At the core of our materials are the Worked Example Pairs (WEPs). Each WEP shows the mathematical work and dialogue of two hypothetical students as they attempt to solve one or more algebra problems. Our curriculum contains four different types of WEPs, with the types varying in what is being compared and the instructional goal of the comparison.

- Which is better? WEPs show the same problem solved using two different correct strategies, with the goal of understanding when and why one strategy is more efficient or easier than the other.
- Which is correct? WEPs show the same problem solved with a correct and incorrect strategy, with the goal of understanding and avoiding common errors.
- Why does it work? WEPs show the same problem solved with two different correct strategies, but with the goal of illuminating the conceptual rationale in one strategy that is less apparent in the other strategy.
- How do they differ? WEPs show two different problems solved in related ways, with an interest in illustrating what the relationship between problems and answers of the two problems reveals about an underlying mathematical concept.

Student Worksheets

In conjunction with the WEPs, we have also created worksheets to facilitate class discussion about multiple solution methods.

Each CEMS worksheet features a question or prompt at the top of the page that is related to the WEP being taught. The worksheets are designed to follow a think-pair-share approach where students first think independently about the worksheet prompt, then turn and share ideas with a partner, and finally engage in whole-class discussion.

At the end of the lesson, the class discusses the 'Big Idea', or main takeaway point that our team has developed for the WEP and the last prompt on the worksheet asks for students to write the Big Idea using their own words.

*Click on the links below to view our CEMS materials.

- Topic 1: Linear Equations
- Topic 2: Functions & Graphing Linear Equations
- Topic 3: Solving Systems of Linear Equations
- Topic 4: Polynomials & Factoring

Poll Results

<http://www.easypolls.net/poll.html?p=5d8ce5c1e4b00d4cf40b6824>

<http://www.easypolls.net/poll.html?p=5d8ce744e4b00d4cf40b6826>

<http://www.easypolls.net/poll.html?p=5d8ce8c7e4b00d4cf40b6828>

We Often Learn Through Comparison

Google - Pixel 3 64GB - Just Black (Verizon)	Samsung - Galaxy Note10+ 256GB - Aura Black (Verizon)	Apple - iPhone XS Max 256GB - Gold (Verizon)	Motorola - moto z* - Flash Gray (Verizon)
Model: G020463-US MSP: 63232045 \$20.83/mo. for 24 mos., O/N APR ★★★★★ (592) See Details Save	Model: SM-N970U MSP: 63232045 \$20.84/mo. for 24 mos., O/N APR ★★★★★ (1,691) Color: Gold See Details Save	Model: M02T178601 MSP: 63232042 \$20.84/mo. for 24 mos., O/N APR ★★★★★ (129) See Details Save	
Carrier Verizon	Verizon	Verizon	Verizon
Wireless Technology 4G LTE	2G, 3G, 4G, 4G LTE	4G LTE	2G, 3G, 4G, 4G LTE
Operating System Android 9.0 Pie	Android 9.0 Pie	Apple iOS 12	Android 9.0 Pie
Usage Time(s) Standby (Always On Display Off): up to 384 hrs; Mixed usage (Always On Display Off): up to 25 hrs	Up to 35 hrs	Talk: up to 1500 min; Playback (wireless video): up to 15 hrs; Playback (wireless audio): up to 65 hrs; Active online usage: up to 13 hrs	-
Internal Memory 64 gigabytes	256 gigabytes	256 gigabytes	128 gigabytes
Screen Size 5.5 inches	6.8 inches	6.5 inches	6.4 inches
Water Resistant Yes	Yes	Yes	No

Babies do too!
Learn categories



Learn words



“Comparison is one of the most integral components of human thought”

(Goldstone, Day & Son, 2010, p. 103)

Comparison is a “Best Practice” in Mathematics Instruction

- Share and compare solution strategies core to reform pedagogy in many countries (Australian Education Ministers, 2006; Brophy, 1999; Kultusministerkonferenz, 2004; NCTM, 2014; Singapore Ministry of Education, 2006; Treffers, 1991)
- Expert teachers use this approach (Lampert, 1990; Richland, Zur & Holyoak, 2007; Shimizu, 1999)

The Essence of Our Approach

Student learning of math, and attitudes toward math, can be improved through the use of:

Comparison



Discussion



Research

- Over 15 years of research supporting the development and testing of this approach
- Several large grants totaling over \$5 million from the US Department of Education and the National Science Foundation
- 30+ publications in academic and teacher journals




Benefits of Compare & Discuss

- Comparing and discussing multiple strategies improves students'
 - Problem-solving accuracy
 - Flexibility: Knowing multiple strategies and when to use them
 - Understanding of key concepts and strategies

EDUCATOR'S PRACTICE GUIDE WHAT WORKS CLEARINGHOUSE


Improving Mathematical Problem Solving in Grades 4 Through 8



Recommendation 4.
Expose students to multiple problem-solving strategies.

1. Provide instruction in multiple strategies.
2. Provide opportunities for students to compare multiple strategies in worked examples.
3. Ask students to generate and share multiple strategies for solving a problem.

NCEE 2012-4055
U.S. DEPARTMENT OF EDUCATION



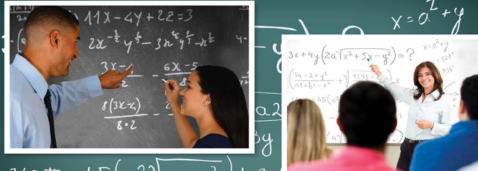
Recommended Teaching Strategy

ies.ed.gov/ncee/wwc/PracticeGuides

EDUCATOR'S PRACTICE GUIDE
A set of recommendations to address challenges in classrooms and schools


WHAT WORKS CLEARINGHOUSE™

Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students



Recommendation 3.
Teach students to intentionally choose from alternative algebraic strategies when solving problems.

NCEE 2015-4010
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Recommended Teaching Strategy

Need to Help Teachers Use Compare & Discuss More Frequently and Effectively

- Comparing strategies rarely done in textbook lessons on **Algebra**
- Only 1% of examples in a US Algebra I textbook included multiple strategies for solving *the same problem*, and comparison was not supported.

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Compare & Discuss: Worked Example Pairs (WEPs)

- Side-by-side comparison of solved problems
- Shows hypothetical students' work and dialogue explaining process
- Includes discussion questions and prompts

Which is better? Topic 2.6

Riley and Gloria were asked to graph the equation $3x - 2y = 6$.

Riley's "x- and y-intercept" way

$3x - 2y = 6$
 $3x - 2(0) = 6$
 $3x = 6$
 $x = 2$
 x-intercept: $(2, 0)$

Then I found the y-intercept by plugging in 0 for x.

$3(0) - 2y = 6$
 $-2y = 6$
 $y = -3$
 y-intercept: $(0, -3)$

I labeled the intercepts and connected them.

Gloria's "slope-intercept" way

$3x - 2y = 6$
 $-2y = -3x + 6$
 $y = \frac{3}{2}x - 3$

I solved for y to put the equation in $y = mx + b$ form.

I graphed the y-intercept of 2. I then used the slope over run to get more points.

I connected the points to get the line.

How did Riley graph the line? Why did Gloria solve the equation for y as a first step?
 Which method is better?

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Use of Compare and Discuss in Typical Algebra Classrooms is Infrequent

Instructional Practice	% of Algebra Lessons
Exposed students to multiple strategies	20
Multiple strategies were compared for at least a 1.5-minute continuous block	1
Engaged in partner/small group work for at least a 1-minute continuous block	29
Had a whole-class discussion for at least a 1.5-minute continuous block	6

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Our Supplemental Compare & Discuss Curriculum for Algebra I

- Accessible online at my.vanderbilt.edu/cems
- Resources tab

Compare & Discuss Problems

Topic 1: Linear Equations



- Materials for each lesson:
 - Teacher Guide for planning
 - Worked-example pair
 - Graphic organizer for student discussion
 - Big Idea
 - (See handout for sample materials)
- 7-9 lessons per topic

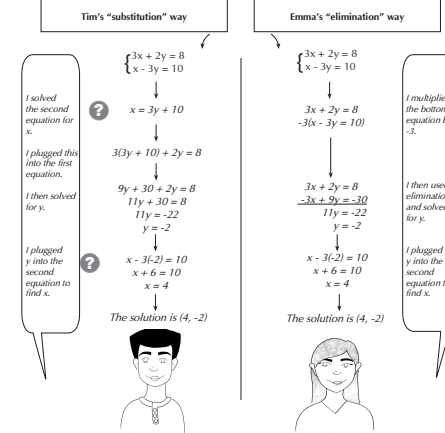
Video Guiding Questions

- In what ways is this lesson leveraging the power of comparison?
 - How do the **materials** support **comparison**?
 - How does the **teacher** facilitate **comparison**?
- See Handout for Topic 3.5:
 - Worked Example Pair
 - Discussion Connections graphic organizer
 - Summary of Big Idea
- See Handout 2 for space to record your ideas. Will use Think-Pair-Share routine because teachers use this routine with our materials.

Video (Using Topic 3.5 Which is Better?)

Tim and Emma were asked to solve the linear system

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$



? Why did Tim choose to plug $y = -2$ into the second equation to find x instead of the first equation?

→ Which method is better? What are some advantages of Tim's "substitution" way? Of Emma's "elimination" way?

Discussion: Think-Pair-Share

- THINK: 1 minute to finish jotting down your thoughts to the prompts on handout
- PAIR: 5 minutes to pair with another participant to discuss your responses to the questions. (Groups of 3 if needed)
- SHARE: 3-4 groups share out given our limited time

PROMPTS reminder:

- In what ways is this lesson leveraging the power of comparison?
 - How do the **materials** support **comparison**?
 - How does the **teacher** facilitate **comparison**?

Your Observations

oSHARE: In what ways is this lesson leveraging the power of comparison?

- How do the **materials** support **comparison**?
 1. X
- How does the **teacher** facilitate **comparison**?
 1. X

How do the materials support comparison?

1. Present two different strategies for solving the same problem.

2. Presented as students' solutions to encourage critical reflection.

3. Make both examples visible and clear; present side-by-side.

Tim and Emma were asked to solve the linear system

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

Tim's "substitution" way

I solved the second equation for x.

$$\begin{aligned} x - 3y &= 10 \\ x &= 3y + 10 \end{aligned}$$

I plugged this into the first equation.

$$3(3y + 10) + 2y = 8$$

$$9y + 30 + 2y = 8$$

$$11y + 30 = 8$$

$$11y = -22$$


$$y = -2$$

I then solved for y.

I plugged y into the second equation to find x.

$$\begin{aligned} x - 3(-2) &= 10 \\ x + 6 &= 10 \\ x &= 4 \end{aligned}$$

The solution is (4, -2)



Emma's "elimination" way

I multiplied the bottom equation by -3.

$$\begin{aligned} 3x + 2y &= 8 \\ -3(x - 3y) &= -3(10) \end{aligned}$$

$$\begin{aligned} 3x + 2y &= 8 \\ -3x + 9y &= -30 \end{aligned}$$


I then used elimination and solved for y.

$$\begin{aligned} 3x + 2y &= 8 \\ -3x + 9y &= -30 \\ \hline 11y &= -22 \\ y &= -2 \end{aligned}$$

I plugged y into the second equation to find x.

$$\begin{aligned} x - 3(-2) &= 10 \\ x + 6 &= 10 \\ x &= 4 \end{aligned}$$

The solution is (4, -2)



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Why did Tim choose to plug $y = -2$ into the second equation to find x instead of the first equation?
 Which method is better? What are some advantages of Tim's "substitution" way? Of Emma's "elimination" way?

4. Prompts for students to:
- A. Understand each strategy.
 - B. Compare strategies to identify pros and cons.

How does the teacher facilitate comparison?

1. Prepare to compare: Take time for students to understand each strategy

2. Make Comparisons

1. Ask students to explain similarities and differences.

1. Mark or list them.

2. Push students to reflect on a key point about the comparison.

Prepare to Compare

- What is the problem asking?
- What is happening in the first method?
- What is happening in the second method?

Make Comparisons

- What are the similarities and differences between the two methods?
 - Which method is better?

Big Idea

- Multiple techniques make comparing strategies more effective, including side-by-side presentation of the strategies and prompting students to identify similarities and differences and pros and cons of the strategies.

Big Idea. Write what you think is the big idea of this video example and discussion, in your own words.

Share. After reviewing together, summarize the ideas we agreed on on your handout.

Video Guiding Questions

● In what ways is this lesson leveraging the power of comparison? What new ideas do you notice?

○ How does the **teacher** facilitate **comparison**?

● See Topic 1.7 worked example pair

● Record ideas on handout 2

Video 2 (Using Topic 1.7 Which is Better?)

Riley and Gloria were asked to solve $5(n + 6) = 2(n + 6) + 6$.

Riley's "distribute first" way

$$5(n + 6) = 2(n + 6) + 6$$

$$5n + 30 = 2n + 12 + 6$$

$$\begin{array}{r} 5n + 30 = 2n + 18 \\ -2n \quad -2n \end{array}$$

$$\begin{array}{r} 3n + 30 = 18 \\ -30 \quad -30 \end{array}$$

$$\begin{array}{r} 3n = -12 \\ 3 \quad 3 \end{array}$$

$$n = -4$$

Gloria's "composite variable" way

$$5(n + 6) = 2(n + 6) + 6$$

$$\begin{array}{r} 5(n + 6) = 2(n + 6) + 6 \\ -2(n + 6) \quad -2(n + 6) \end{array}$$

$$\begin{array}{r} 3(n + 6) = 6 \\ 3 \quad 3 \end{array}$$

$$\begin{array}{r} n + 6 = 2 \\ -6 \quad -6 \end{array}$$

$$n = -4$$

First, I distributed.

Then I moved the variable to one side of the equation.

I subtracted from both sides.

I divided by 3.

Here's my answer.

First, I subtracted the quantity $2(n + 6)$ from both sides.

Then I divided by 3.

I subtracted from both sides.

Here's my answer.

❓ How did Riley and Gloria solve the equation?

➡ Which method is better? What are some important differences between Riley's "distribute first" method and Gloria's "composite variable" method?

Discussion: Think-Pair-Share

- THINK: 1 minute to finish jotting down your thoughts
- PAIR: 3 minutes to pair with another participant to discuss your responses to the question.
- SHARE: 2-3 groups share out given our limited time

PROMPTS:

■ In what ways is this lesson leveraging the power of comparison?

What new ideas do you notice?

○ How does the **teacher** facilitate **comparison**?

How does the teacher facilitate comparison?

2. Make Comparisons

1. Ask students to explain similarities and differences.

1. Use color coding to mark, write notes.
2. Doesn't stop at 1-2 similarities or differences.

2. Push students to reflect on a key point about the comparison.

Make Comparisons

- What are the similarities and differences between the two methods?
 - Which method is better?

Types of Comparisons

See your handout

Type 1: Which is better?

- Compare two correct strategies and reflect on when one strategy is better than another
- Examples Topic 3.5 and Topic 1.7 (video)

Type 2: Why does it work?

- Compare two correct strategies to better understand why the teacher-taught strategy works
- Example Topic 2.1

Type 3: Which is correct?

- Compare a correct and incorrect strategy to understand why common mistakes are incorrect and to increase use of correct strategies.
- Example Topic 4.5

Why Does It Work?

Compare two correct strategies to better understand why the teacher-taught strategy works

Why does it work?

Riley and Gloria were given the set of ordered pairs $\{(-3, 6), (2, 5), (3, 1), (2, 4), (5, 1)\}$, and asked to determine if the relation is a function.

Riley's "make a table" way

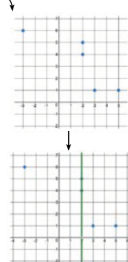
x (domain)	y (range)
-3	6
2	5
2	4
3	1
5	1

x (domain)	y (range)
-3	6
2	5
2	4
3	1
5	1

Not a function



Gloria's "graph and vertical line test" way



Not a function



I made a table.
I saw that 2 in the domain is paired with both a 5 and a 4 in the range.
This means the relation is not a function.

I graphed the ordered pairs.
I found a vertical line that intersected two of the points.
This means the relation is not a function.

How did Riley determine if the relation was a function? How did Gloria determine if the relation was a function?

Why do both methods work? Why does the vertical line test tell us the same thing as the table of values?

Which is correct?

Topic 4.5

Layla and Riley were asked to use factoring to solve the equation $a^2 + 5a - 6 = -12$.

Layla's "set equal to 0" way

$$\begin{aligned}
 a^2 + 5a - 6 &= -12 \\
 a^2 + 5a + 6 &= 0 \\
 (a + 2)(a + 3) &= 0 \\
 a + 2 = 0 \text{ or } a + 3 = 0 \\
 a = -2 \text{ or } a = -3
 \end{aligned}$$



Riley's "factor first" way

$$\begin{aligned}
 a^2 + 5a - 6 &= -12 \\
 (a + 6)(a - 1) &= -12 \\
 a + 6 = 6 \text{ or } a - 1 = -2 \\
 a = 0 \text{ or } a = -1
 \end{aligned}$$



Which is correct?

Compare a correct and incorrect strategy to understand why common mistakes are incorrect and to increase use of correct strategies.

First, I set the equation equal to zero by adding 12 to both sides. Then, I factored. I solved the equations to get my answers.

First, I factored. Since 6 times -2 is -12, I set the first part equal to 6 and the second part equal to -2. Then I solved the equations to get my answers.

How could you check to see if Layla or Riley's solutions are correct?

Which method is correct, Layla's "set equal to 0" method or Riley's "factor first" method?

How do the materials support discussion?

Graphic organizer to support a) Think – pair – share routine

b) Students summarizing the big idea in own words

Discuss Connections

Is there a situation where substitution would be better than elimination, or vice versa?

<p>Think, Pair. First, think about the question(s) above independently. Then, get with a partner and discuss your answers. After talking with your partner, what is your answer?</p> <p style="text-align: center;">Think</p>	<p style="text-align: center;">Pair</p>
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Share. After reviewing the worksheet as a class, summarize the answer(s) your class agrees on. Was this different from your original response?

Prepare to Discuss (Think, Pair)

Is there a situation where substitution would be better than elimination, or vice versa?

Discuss Connections (Share)

If one of the equations has a variable with a coefficient of 1, that equation is easy to rearrange, so substitution might be better. If the same variable in both equations has the same or opposite coefficients, then elimination might be better.

Identify the Big Idea

Can you always use either substitution or elimination? Which is better? When solving a system of linear equations, substitution and elimination are both correct methods that will give you the same answer. Substitution might be easier if the variable has a coefficient of 1.

Teacher Guide includes teacher questions and a potential student answer

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Video Guiding Questions

- In what ways is this lesson leveraging the power of discussion?
 - How does the **teacher** facilitate **discussion**?
- See Handouts for Topic 3.5:
 - Worked Example Pair
 - Discussion Connections graphic organizer
 - Summary of Big Idea

How do the materials support discussion?

Slide summarizing a Big Idea that should emerge from discussion.

Tim and Emma were asked to solve the linear system

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

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Video of Discussion (Using Topic 3.5 Which is Better?)

Discussion: Think-Pair-Share

- THINK: 1 minute to finish jotting down your thoughts
- PAIR: 5 minutes to pair with another participant to discuss your responses to the questions
- SHARE: 3-4 groups share out given our limited time

PROMPTS:

- In what ways is this lesson leveraging the power of discussion?
 - How does the **teacher** facilitate **discussion**?
 - X

Leveraging the Power of Discussion

How can teachers support discussion?

1. During the discussion:
 - Asking open-ended questions (e.g., "Why do you think that's true?")
 - Re-voicing and summarizing contributions
 - Hearing from many voices
 - Holding participants accountable for listening to others: "Do you agree or disagree with Morgan? Why?",

Big Idea

THINK: *On a notecard:*

- BIG IDEA: Write what you think is the big idea of this presentation, in your own words.
- USE IT: Write 1-3 things you learned from today that you plan to use in your own instruction.

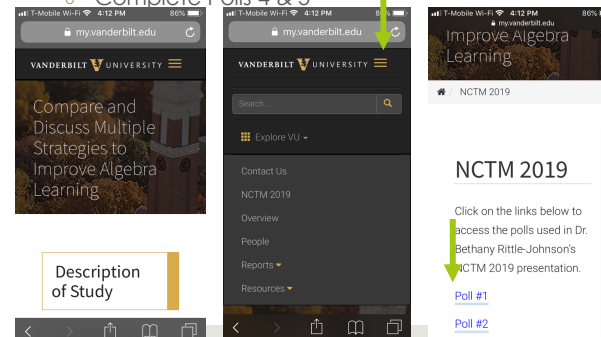
PAIR: Share with your partner

SHARE: Share with me by dropping off your notecard on front table

Complete Polls 4 & 5 online

Audience Exit Polls 4 & 5

- Go to: my.vanderbilt.edu/cems/
 - Select NCTM 2019 from menu
 - Complete Polls 4 & 5



Poll Results

- <http://www.easypolls.net/poll.html?p=5d8d0dc8e4b00d4af40b68ab>
- <http://www.easypolls.net/poll.html?p=5d8d0dece4b00d4af40b68ac>
- Side note: We are hoping to work with Integrated Math I teachers in Metro Nashville Public Schools next year! Let me know if you are an MNPS teacher who might be interested in participating if the project works out.

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Select References

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Acknowledgements

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